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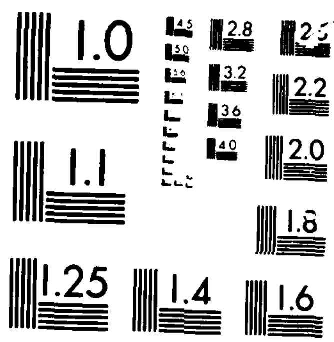

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20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

The research activities supported by this grant are summarized. Theoretical and computational research is discussed.

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SOME INVESTIGATIONS INTO VARIABLE MESHES FOR NUMERICAL SOLUTION OF PARTIAL DIFFERENTIAL EQUATIONS

TODD F. DUPONT

22 MAY 1986

U. S. ARMY RESEARCH OFFICE

DAAG29-82-K-0157

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SOME INVESTIGATIONS INTO VARIABLE MESHES FOR NUMERICAL SOLUTION OF PARTIAL DIFFERENTIAL EQUATIONS

Final Report of Army Research Contract DAAG29-82-K-0157

Todd F. Dupont
University of Chicago
22 May 1986

This study involved investigation of numerical methods for the approximate solution of partial differential equations. Specifically, the following two topics were investigated:

- Spatial meshes that change with time
- Space-time finite elements.

The work involved both theoretical investigations and computational experiments. Some of these results have been presented in meetings, some are incomplete, and some are still waiting to be written up.

1. Spatial Meshes That Change with Time

Several aspects of the problems associated with using moving meshes in the numerical solution of evolution equations were studied from a theoretical viewpoint. Also, a moving mesh code was produced for parabolic equations in two space dimensions; this program is discussed in Section 2 since it uses space time finite elements in its time discretization.

1.1 Moving Finite Elements in Polygonal Regions

A version of the Moving Finite Element Method [MM] was examined for parabolic problems (convection-diffusion problems) and some basic results were established for the case in which the domain is polygonal.

As in the one-dimensional case examined in [D1], a major problem with the MFE is establishing the existence of the solution, since this requires a simultaneous proof of stability. One way to view the mesh movement is think of it as being caused by a time-dependent transformation of the spatial domain onto itself. In the one-dimensional context the transformation of the domain was shown to be invertible at each time level by showing that it was strictly monotone; this involved a continuity argument. In the two dimensional case the examination of the transformation is more complicated. Degree theory, along with a continuity argument, played a central role in showing the nonsingularity of the transformation.

In the case of discrete time the continuation argument breaks down, but the degree theory still is useful in giving an easily computed (local) test to see if the transformation generating the mesh movement is globally invertible. This allows the program that is doing the calculation to be able to see if the mesh has folded over on itself or has mapped part of the mesh outside the domain.

Once the stability and existence of the numerical solution are established, the usual a priori error estimates in the "natural" norm can be proved using approximation theory and the results in [D1]. The norm involved here is the sum of three simpler norms; perhaps the strongest norm is the L2 in time of the H1 in space norm and the error estimates are optimal order in this norm.

These results were presented at The 1983 Dundee Conference on Numerical Analysis, held in Scotland at the University of Dundee in June, 1983. They should have appeared in the proceedings of that conference, but the manuscript is still unfinished.

1.2 Fixed-in-time Nonunifrom Mesh for First Order Hyperbolics

The partial differential equations for which moving meshes seem most attractive involve a convective term that causes effects to flow from one part of the region to another. The meshes produced by moving mesh methods are essentially always nonuniform. Since the methods I was looking at were for the most part based on Galerkin methods it seemed important to understand the convergence properties of such methods on nonuniform meshes.

For stable boundary conditions convergence of such methods was established in [SW], but the order of convergence was not optimal from the point of view of approximation theory. In the case of a uniform mesh for piecewise linear functions, optimal order convergence was proved in [D2] and later results by Wahlbin even showed superconvergence for smooth splines on uniform meshes in the periodic case. If the approximating function is piecewise linear and there are only a finite number of mesh changes the Layton has shown that there is only a loss of 1/2 in the order of convergence.

Both positive and negative results were derived, as a part of this study, for Galerkin methods for piecewise linear functions over a variable mesh. First, if there are at most a finite number of changes in the mesh size then the order of convergence in the L2 norm (in space) is two; this is the optimal order and generalizes the corresponding result of [D2] to allow mesh changes and a nonperiodic case. Second, for a simple mesh that is nonconstant it can be shown that the order of convergence is down by one from what approximation theory says is possible; this mesh has intervals that alternate between size h and size h/2. This negative result is similar in spirit and proof to the Hermite cubic result of [D2].

The importance of the negative result in the context of moving mesh Galerkin methods is that the mesh must come quite close to tracking the underlying convective flow or there must be diffusive terms in the differential equation or there is likely to be unnatural behavior in the approximate solution. If one is willing to generalize this feeling to systems, it seems to say that the mesh for each component of the solution needs to move at the speed of effects in that component. This is serious, because it gives strong indication that there needs to be more than one moving mesh on many interesting systems, and this, in turn, adds interest to the result described in 1.4 below.

These results were presented at the Finite Element Circus held at Cornell University in May 1983. They have not been written up for publication, but they should be.

A result on optimal order methods that do allow variable meshes was reworked and written up for publication during the period of this study. These methods [DW] are more complicated, but allow for stagnation in the flow, something that is difficult for some methods for hyperbolic equations. The method of this paper has not been looked at for moving meshes.

1.3 A Posteriori Error Estimates for Nonlinear Parabolic Equations

In any adaptive methods for the numerical solution of differential equations a posteriori error estimations are crucial, because without them one would not know if the quality of the answer is acceptable. One good way to build moving mesh methods would be to use local error estimators to see which areas of the domain were mesh rich and which were mesh poor. As is illustrated by the work in [BB] the a posteriori error estimates for linear parabolic Galerkin methods are in good shape.

As a first step to getting such error estimators for nonlinear equations using moving mesh methods, an experimental study was undertaken with a graduate student, Patrick Getz. This study took longer than expected and it only produced programs for parabolic and hyperbolic Galerkin methods in one space dimension using fixed mesh methods. There were, however, unexpected results to be considered in the nonlinear parabolic case.

In the case of linear parabolic equations it is known that at positive times Galerkin methods produce optimal order approximations (from the point of view of approximation theory) even if the initial data for the problem is very rough. This result is seen from both the a priori [LR] and a posteriori [BB] error estimates. However, it is not the case that the nonlinear theory is parallel to the linear in this regard.

As an experimental test of whether this optimal order convergence at later time holds in the case of a nonlinear equation, we used a Galerkin method based on continuous quadratics for a nonlinear parabolic equation in divergence form. The initial data were taken to have a singularity in the interior of the region of the type that sin(ln(abs(x))) has at the origin. This function belongs to any L2-based Sobolev space of index less than 1/2. The Galerkin method was started with the L2 projection and the error was measured at a fixed positive time in the L2 norm. As a test the code was used on a linear problem and, as the theory predicts, the error went to zero as the cube of the mesh length.

For a highly nonlinear problem, over a wide range of the size of the mesh parameter, the error went to zero as the mesh size to the 1.5 power. This is a striking difference from the linear case, so we worked on an explanation. Eventually, it was found that the loss of convergence rate can be viewed as coming from an instability in the linearized equation in areas where the solution is very rough. It was found that the 1.5 power was not the asymptotic rate, but that the asymptotic rate did not obtain until the mesh was so fine that rounding error using 64-bit words was the size of the discretization error. The calculations made it easy to examine the quality of the a posteriori error indicator used in the case of the linear equations. It was really very good for the heat equation, but seemed to be far off the mark in the nonlinear case.

We drew three conclusions from these experiments. First, it may be that the

nonlinear equation has the same asymptotic convergence rate as the linear one. Second, it does not matter, since it is not seen until the mesh is extremely small. Third, the a posteriori error indicators that can be used for the linear equation with good success are not useful in a strongly nonlinear problem with rough initial data. Mr. Getz drew one additional conclusion from these experiments, that he would like to work on some other kind of mathematics.

These results were presented the Conference on Numerical Analysis, Woudschoten in Zeist, the Netherlands, September, 1983. They have not been, and probably will not be, formally written up.

1.4 Communication Between Moving Irregular Grids

As a part of the preliminary design of the moving mesh program discussed below, Randolph Bank and I examined the complexity of the communication between two different moving grids. For the reasons discussed in Section 1.3 above, this may be a necessary part of such programs.

In a model case, with assumptions that are restrictive but not ridiculous, we were able to see that the work associated in setting up the links between grids is proportional to the total number of grid points. We looked at triangular meshes on a planar region that were continuously deformed from one step to the next. Specifically, we showed that if the movement of the meshes on each time step is such that no point in the domain moves through more than a fixed number of triangles in either mesh then (using data structures that take space proportional to the number of mesh triangles) the work to locate the vertex of each triangle in one mesh in the other mesh is proportional to the number of vertices. (Note that to good approximation the number of triangles is half the number of vertices.) In particular, if the meshes are quasi-uniform and the movement is smooth, then a "bounded Courant number" condition gives the claimed complexity bound.

We were surprised at the simplicity of this result and the almost trivial nature of its proof. It indicates that the communication between different grids is not nearly as hard as it seems to be generally thought to be, and it is certainly not as hard as we had thought it was.

1.5 New Work Based on the Results Obtained

In the period after this contract ended some additional work has been done based on the results obtained thus far. This work has been carried out with a visitor to the University of Chicago, Che Sun, from China. Together we have proved some new a priori error estimates for moving mesh Galerkin methods for nonlinear parabolic equations. These estimates have different flavor from those that were proved in [D1] in the sense that the more recent results do not constrain the functions that can be used to interpolate the solution when applying the bounds. We also proved that in a one space dimensional problem that optimal order L2 estimates hold for moving mesh methods, provided the mesh moves in a regular way. (It was shown in [D1] that if the mesh movement is very irregular, then such methods may converge to the wrong answer.)

2. Space Time Finite Elements

There are two ways to view space-time finite elements. The first is that they provide a useful way to think about moving meshes, guiding the choice of time discretization. The second is that they provide a means of constructing schemes in which the time step is spatially varying. The program discussed in Section 2.1 is based on the first use of space-time elements and the method discussed in Section 2.2 is motivated by the second.

2.1 Prism Element Methods and Generalizations

R. E. Bank, with some help from me, produced a program for use in testing mesh moving strategies for convection diffusion equations. In it the movement of the mesh, the selection of the time step, the estimation of the error, and the solution of the parabolic Galerkin equations are all separate steps. The program is based in large part on some of the elliptic and parabolic Galerkin programs that Professor Bank has used in other contexts. So far this program has not been used in the way it was intended; it has been subjected to some testing but not used to test mesh selection rules.

The program is based on a space-time finite element discretization that is a generalization to the context of moving meshes of the "theta-weighted" discretization family; in particular the fixed mesh version of this code includes both Crank-Nicolson and backward Euler time discretizations. The space-time finite elements include triangular based prisms and degenerate versions of these prisms in which the triangle on the top or bottom (but not both) has collapsed to a point or a line. These space-time elements are useful for adding and removing freedom as well as moving triangles from one part of the domain to another.

The function space at each time level consists of continuous piecewise linear functions over a triangular mesh.

2.2 A General Space-Time Method Finite Element Method

A general space-time mesh method that is in some ways analogous to the usual Crank-Nicolson method has been studied. The test space is the time derivative of the trial space, and there is no constraint that the space be linear in time. (If the space mesh is fixed and the trial space is viewed a being linear between time levels, then Crank-Nicolson is the resulting scheme.)

A restricted stability result has been proved. It gives an appropriate L2 bound at time levels for which a full bend is possible, in the sense that all functions in the space can be redefined above the level so as to be independent of time. However, a general L2 stability result has not been established. The times at which a full bend is possible are the marching times of this algorithm; the solution process between such times is global and is not easily broken down into smaller calculations.

In one space dimension the method has reasonable algorithmic properties, in the important case in which the roughness that causes the need fo the locally refined time step is induced on the boundary. Although it is not as

simple to program as the usual finite difference methods, the work estimate in terms of the number of unknowns computed is good. In multiple dimensions the method seems much more complicated to actually compute and additional work is needed to determine if it is a viable method.

No experiments with this method were carried out as a part of this contract.

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